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Harmonic oscillator interacting with conical singularities

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Abstract. We study the quantum dynamics of a particle interacting harmonically with conical singularities that physically correspond to either a cosmic string, a global monopole, a magnetic flux string or a screw dislocation, by solving the corresponding Schrödinger equations. Exact expressions for the energy eigenfunctions and eigenvalues are found.

1. Introduction

The study of quantum systems with non-standard boundary conditions has been an exciting research field in recent years. For example, many authors have studied the quantum dynamics of a particle interacting with a wedge. Recently, Crandall [1] and DeWitt-Morette [2] evaluated the propagator for a free particle interacting with a rational wedge. Cheng [3, 4] studied the quantum dynamics of a particle interacting harmonically with a wedge. In another paper, Zhu [5] investigated the dynamics of a harmonic oscillator in the presence of a magnetic flux that passes through the plane at the equilibrium point of the motion. In this work we analyse the quantum dynamics of a particle subjected to non-trivial boundary conditions, which are imposed by conical singularities in different physical contexts.

The purpose of this paper is to investigate the quantum dynamics of a single particle interacting harmonically with a topological defect. These defects are characterized by a spacetime metric with a Riemann–Christoffel curvature tensor and/or torsion tensor which is null everywhere except on the defects, that is, by the conical type of curvature or torsion singularities [6, 7]. Some examples of curvature conical singularity topological defects are cosmic strings [8] and global monopoles [9]. An example of torsion conical singularity is the cosmic dislocation [10]. These defects appear naturally in gauge theories with spontaneous symmetry breaking and may have played important roles in the formation of the large-scale structure of the universe [8]. They are also important in the context of the geometrical theory of defects in solids [11, 12, 17].

A quantum particle is considered in each of the following background spacetimes: a cosmic string, a magnetic flux string, a global monopole and a cosmic dislocation. We consider a harmonic interaction potential between the particle and the defect. This model is a pedagogical approach for the study of the influence of conical singularities in the quantum dynamics of a single particle. It also is motivated by the possibility of using the vibrational spectroscopy of diatomic molecules as an (approximate) probe for topological defects in the

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cosmos. Probes such as these have been suggested earlier, for example, using Rydberg atoms [13], Lamb shifts [14] and the energy shifts of hydrogen atoms [15].

This paper is organized as follows. In section 2 we obtain exact solutions of the Schrödinger equation (SE) for a two-dimensional harmonic oscillator interacting with a cosmic string; in section 3 with a magnetic flux string; in section 4 we obtain exact solutions of the SE for a three-dimensional harmonic oscillator interacting with a global monopole. In section 5 we analyse a two-dimensional harmonic oscillator interacting with a cosmic dislocation and, finally, in section 6 we summarize our main results.

2. Harmonic oscillator in the presence of a cosmic string

The line element corresponding to the cosmic string spacetime is given in cylindrical coordinates by [8]

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}\rho^2 - \alpha^2\rho^2\,\mathrm{d}\varphi^2 - \mathrm{d}z^2 \tag{1}$$

where $\rho \geqslant 0$ and $0 \leqslant \varphi \leqslant 2\pi$, the parameter $\alpha = 1 - 4G\mu$ and μ is the linear mass density. This metric has a cone-like singularity at $\rho = 0$. In other words, the curvature tensor of the metric (1), considered as a distribution, is of the form

$$R_{12}^{12} = 2\pi\frac{\alpha - 1}{\alpha}\delta^2(\rho) \tag{2}$$

where $\delta^2(\rho)$ is the two-dimensional Dirac δ-function. This fact characterizes a two-dimensional conical singularity.

Let us consider a non-relativistic quantum particle embedded in a classical background field. Its behaviour is described by the Schrödinger equation [16]

$$\mathrm{i}\frac{\partial\Phi(q,t)}{\partial t} = -\frac{1}{2M}\Delta\Phi(q,t) + V(q) \tag{3}$$

where m is the mass of the particle and we choose units such that $\hbar = 1$. The symbol Δ is the Laplace–Beltrami operator

$$\Delta = \frac{1}{\sqrt{g}}\partial_i\left(g^{ij}\sqrt{g}\partial_j\right)$$

$g = \det|g_{ij}|$ and the Latin indices run over the space coordinates only. The Schrödinger equation in the metric (1) is of the form

$$\mathrm{i}\frac{\partial\Psi}{\partial t} = -\frac{1}{2M}\left[\frac{1}{\rho}\partial_\rho(\rho\partial_\rho) + \partial_z^2 + \frac{1}{\alpha^2\rho^2}\partial_\varphi^2\right]\Psi + V(\rho)\Psi \tag{4}$$

where $V(\rho)$ is a cylindrically symmetric interaction potential assumed to be

$$V(\rho) = \tfrac{1}{2}m\omega^2\rho^2. \tag{5}$$

Using the change of variables $\sigma = \gamma\rho^2$ and assuming for the eigenfunction the form

$$\Psi = \mathrm{e}^{-\mathrm{i}Et+\mathrm{i}\ell\varphi+\mathrm{i}kz}\mathrm{e}^{-\sigma/2}\sigma^{|\ell|/2\alpha}R(\sigma) \tag{6}$$

which satisfies the usual asymptotic requirements and finiteness at the origin for a bound state, we have

$$\sigma\frac{\mathrm{d}^2R}{\mathrm{d}\sigma^2} + \left[\left(1 + \frac{|\ell|}{\alpha}\right) - \sigma\right]\frac{\mathrm{d}R}{\mathrm{d}\sigma} - \left[\left(1 + \frac{|\ell|}{\alpha}\right) - \frac{A}{2\gamma}\right]R \tag{7}$$

where $\gamma^2 = M^2\omega^2$ and $A = -2ME + k^2$.

We find that the solution of equation (7) is the degenerated hypergeometric function

$$R = F\left(a, \frac{|\ell|}{\alpha} + 1; \gamma\rho^2\right) \tag{8}$$

where $a = (1 + |\ell|/\alpha - A/2\gamma)$. In order to have normalization of the wavefunction, the series in (8) must be a polynomial of degree n, therefore

$$a = -n. \tag{9}$$

With this condition, we obtain discrete values for the energy given by

$$E = \omega\left(2n + \frac{|\ell|}{\alpha} + 1\right) + \frac{k^2}{2M} \tag{10}$$

where $n = 0, 1, 2, \ldots$.

The energy eigenfunction is then given by

$$\Psi = C_{n\ell}\, \rho^{|\ell|/2\alpha} \mathrm{e}^{\mathrm{i}kz} \mathrm{e}^{\mathrm{i}\ell\varphi} \mathrm{e}^{-\gamma\rho^2/2}\, {}_1F_1\left(-n, \frac{|\ell|}{\alpha} + 1, \gamma\rho^2\right) \tag{11}$$

where $C_{n\ell}$ is a normalization constant. It is easy to see that the presence of the parameter α breaks the degeneracy of the energy levels. In the limit of $\alpha \to 1$ equation (10) gives the usual two-dimensional harmonic oscillator levels. Our results agree with those found by Cheng [3] and Cheng and da Luz [4] who studied a quantum particle in the presence of a wedge using path integrals. Note that the boundary conditions imposed by the cosmic string are identical to those determined by the wedge, namely

$$\Psi(\rho, 0, z) = \Psi(\rho, 2\pi\alpha, z)$$

for a wedge of dihedral angle $2\pi(1 - \alpha)$.

3. Harmonic oscillator in the presence of a magnetic flux string

Now, we proceed as in the previous case to study a charged harmonic oscillator in the presence of a magnetic flux cosmic string. We consider that the internal magnetic field is of the form

$$A(\rho) = \frac{\Phi}{2\pi\alpha}\frac{1}{\rho}\hat{e}_\varphi \tag{12}$$

where Φ is the magnetic flux of the string. Using minimal coupling, the time-independent Schrödinger equation for the harmonic oscillator is in this case

$$\left[-\frac{1}{2}\left\{\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{\partial^2}{\partial z^2} + \frac{1}{\alpha^2\rho^2}\frac{\partial^2}{\partial\varphi^2}\right\} - \frac{\mathrm{i}q\Phi}{\pi M\alpha^2\rho^2}\frac{\partial}{\partial\varphi} + \frac{\Phi^2}{8\pi^2\alpha^2\rho^2 M} + \frac{1}{2}M\omega^2\rho^2\right]\psi = E\psi \tag{13}$$

where q is the charge of the particle.

By using again the change of variables $\sigma = \gamma\rho^2$ and assuming for the eigenfunction the form

$$\psi = \exp[\mathrm{i}kz + \mathrm{i}\ell\varphi]\mathrm{e}^{-\sigma/2}\sigma^{(|\ell| + q\Phi/2\pi)/2\alpha}u(\sigma) \tag{14}$$

we have

$$\sigma^2\frac{\mathrm{d}^2u}{\mathrm{d}\sigma^2} + \left[1 + \frac{|\ell| + q\Phi/2\pi)}{\alpha} - z\right]\frac{\mathrm{d}u}{\mathrm{d}\sigma} - \left[1 + \frac{\ell + q\Phi/2\pi|}{\alpha} + \frac{C^2 - k^2}{2\gamma}\right]u = 0 \tag{15}$$

where $C^2 = 2ME$ and $\gamma^2 = m^2\omega^2$.

As before, we find that the solution of equation (15) is the degenerate hypergeometric function

$$u = F\left(a, \frac{|\ell + q\Phi/2\pi|}{\alpha} + 1; \gamma\rho^2\right) \tag{16}$$

where $a = \left(1 + \frac{|\ell+q\Phi/2\pi|}{\alpha} - \frac{(k^2-C^2)}{2\gamma}\right)$. Again, in order to have normalization of the wavefunction, the series in (16) must be a polynomial of degree n. Therefore, we must impose

$$a = -n. \tag{17}$$

With this condition we obtain discrete values for the energy levels given by

$$E = \omega\left(2n + \frac{|\ell + q\Phi/2\pi|}{\alpha} + 1\right) + \frac{k^2}{2m} \tag{18}$$

with $n = 0, 1, 2, \ldots$. The energy eigenfunction is given by

$$\psi = C_{n\ell}\,\rho^{(|\ell+q\Phi/2\pi|)/2\alpha}\,\mathrm{e}^{\mathrm{i}kz}\,\mathrm{e}^{\mathrm{i}\ell\varphi}\,\mathrm{e}^{-\gamma\rho^2/2}\,F\left(-n, \frac{|\ell + q\Phi/2\pi|}{\alpha} + 1, \gamma\rho^2\right) \tag{19}$$

where $C_{n\ell}$ is a normalization constant. Note that the inclusion of the magnetic flux Φ breaks the degeneracy of the energy levels further.

4. Harmonic oscillator in the presence of a global monopole

Barriola and Vilenkin [9] have shown that the effects produced by a global monopole in the geometry can be approximately represented by a solid angle deficit in (3 + 1)-dimensional spacetime. The metric of this manifold can be expressed by the line element, in spherical coordinates,

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{\alpha^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right) \tag{20}$$

where the parameter $\alpha^2 = 1 - 8\pi G\nu^2$ is smaller than 1 and depends on the energy scale ν. The area of a sphere of unit radius in this manifold is not 4π but $4\pi\alpha^2$, and the surface $\theta = \pi/2$ presents the geometry of a cone with deficit angle $\lambda = 8\pi G\nu^2$. This is an example of a three-dimensional conical singularity in the spacetime metric. The curvature tensor of the metric (20) is of the form

$$R^{\theta\varphi}_{\theta\varphi} = \frac{\alpha^2 - 1}{\alpha^2} r^{-2}. \tag{21}$$

The time-independent Schrödinger equation in the metric (20) is described by

$$-\frac{1}{2m}\left\{\frac{\alpha^2}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}}{\mathrm{d}r}\right) - \frac{L^2}{r^2}\right\}\psi + \tfrac{1}{2}m\omega^2 r^2\psi = E\psi \tag{22}$$

where L^2 is the angular momentum squared operator.

We assume for the eigenfunction the form

$$\psi = CY^m_\ell(\theta, \psi)R(r) \tag{23}$$

and the change of variables $\sigma = r^2$, obtaining

$$\sigma^2\frac{\mathrm{d}R}{\mathrm{d}\sigma^2} + \frac{3}{2}\sigma\frac{\mathrm{d}R(\sigma)}{\mathrm{d}\sigma} + \left(\tfrac{1}{4}K\sigma - \tfrac{1}{4}\gamma\sigma^2 - \tfrac{1}{4}D\right)R(\sigma) = 0 \tag{24}$$

with $K = 2mE/\alpha^2$, $\gamma = m^2\omega^2/\alpha^2$ and $D = \ell(\ell+1)/\alpha^2$. We find that the solution of equation (24) is the hypergeometric function

$$R(\sigma) = F\left(a, 1 + \frac{\sqrt{\alpha^2 + 4\ell(\ell+1)}}{2\alpha}, \sigma\right) \tag{25}$$

with

$$2a = \gamma - \frac{K}{4\alpha}. \tag{26}$$

In order to have normalization of the wavefunction (25) we impose

$$a = -n. \tag{27}$$

This condition determines the discrete values for the energy to be given by

$$E = \frac{\omega}{\alpha}\left(2n + \frac{\sqrt{\alpha^2 + 4\ell(\ell+1)}}{2\alpha} + 1\right). \tag{28}$$

The energy eigenfunction is given by

$$\psi(r, \theta, \varphi) = C_{\ell n}, Y_\ell^m(\theta, \varphi) F\left(a, 1 + \frac{\sqrt{\alpha^2 + 4\ell(\ell+1)}}{2\alpha}, r^2\right) \tag{29}$$

where $C_{\ell n}$ is a normalization constant.

Note that the parameter α breaks the degeneracy of the energy levels. In the limit $\alpha \to 1$ the energy levels coincide the usual three-dimensional harmonic oscillator levels.

5. Harmonic oscillator in the presence of a screw dislocation

In this section we investigate a harmonic oscillator in a background field of cosmic dislocation. The line element of this defect is given in cylindrical coordinates by

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}\rho^2 + \rho^2\,\mathrm{d}\varphi^2 + (\mathrm{d}z + \beta\,\mathrm{d}\varphi)^2 \tag{30}$$

with $\rho \geqslant 0$ and $0 \leqslant \varphi \leqslant 2\pi$. The parameter β is related to torsion. In the language used in crystallography β is related to the Burgers vector. This metric contains a conical singularity in the torsion tensor. The torsion associated with this defect corresponds to a conical singularity at the origin. The only non-zero component of the torsion tensor in this case is given by the 2-form

$$T^1 = 2\pi\beta\delta^2(\rho)\,\mathrm{d}\rho \wedge \mathrm{d}\phi \tag{31}$$

where $\delta^2(\rho)$ is the two-dimensional delta function in flat space. The three-dimensional geometry of the medium, in this case, is characterized by non-trivial torsion, which is identified with the surface density of the Burgers vector in the classical theory of elasticity. In this way, the Burgers vector can be viewed as a flux of torsion, given by

$$\int_\Sigma T^1 = \oint_S e^1 = 2\pi\beta = b \tag{32}$$

where we adopt the following triad representation (1-form basis) for the metric (30)

$$e^1 = \mathrm{d}z + \beta\,\mathrm{d}\phi \tag{33}$$

$$e^2 = \mathrm{d}\rho \tag{34}$$

$$e^3 = \rho\,\mathrm{d}\phi \tag{35}$$

and the torsion 2-form is related to the triad by

$$T = \mathrm{d}e + \Gamma^{(L)} \wedge e \tag{36}$$

where $\Gamma^{(L)}$ is the Lorentz connection, which is zero for this geometry since there is no curvature involved. This equation leads to the result (31) when we substitute (33)–(35) into it. We write the torsion in tensor notation

$$T^a_{\mu\nu} = \partial_\mu e^a_\nu - \partial_\nu e^a_\mu \tag{37}$$

where the 2-form component $T^a = T^a_{\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$ and the triad component $e^a = e^a_\mu\,\mathrm{d}x^\mu$.

The Schrödinger equation for the harmonic oscillator in the metric (30) is expressed by

$$\left\{-\frac{1}{2m}\left[\frac{\partial^2}{\partial z^2} + \frac{1}{\rho^2}\left(\frac{\partial}{\partial\varphi} - \beta\frac{\partial}{\partial z}\right)^2 + \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}\right]\psi + \tfrac{1}{2}m\omega^2\rho^2\varphi\right\} = E\psi. \tag{38}$$

Using the following ansatz

$$\psi(\rho,\varphi,z) = \exp[\ell\varphi + \mathrm{i}kz]\mathrm{e}^{-\sigma/2}\sigma^{|\ell-\beta k|/2}u(\sigma) \tag{39}$$

where ℓ and k are constants, and $\sigma = \gamma\rho^2$, we have

$$\sigma\frac{\mathrm{d}^2u(\sigma)}{\mathrm{d}\sigma^2} + [1 + |\ell - \beta k| - \sigma]\frac{\mathrm{d}u(\sigma)}{\mathrm{d}\sigma} - \left(\frac{A}{2\sqrt{\gamma}} + 1 + |\ell - \beta k|\right)u(\sigma) \tag{40}$$

where $A = k^2 - 2mE$ and $\gamma = m^2\omega^2$.

We find that the solution of equation (40) is the degenerated hypergeometric function

$$u = F\left(\zeta, |\ell - \beta k| + 1,\ \tfrac{1}{2}(m\omega\rho^2)\right) \tag{41}$$

with $a = |\ell - \beta k| + 1 - A/2\sqrt{\gamma}$. Setting $\zeta = -n$, as before, to make the eigenfunction normalizable we have

$$E = \omega(n + |\ell - \beta k| + 1) + \frac{k^2}{2m}. \tag{42}$$

The eigenfunction is then given by

$$\psi = C_{n\ell}\,\rho^{|\ell-\beta k|}\mathrm{e}^{\mathrm{i}kz}\mathrm{e}^{\mathrm{i}\ell\varphi}c^{-\gamma\rho^2/2}F(-n, |\ell - \beta k| + 1, m\omega\rho^2) \tag{43}$$

where $C_{n,e}$ is a normalization constant.

The conical singularity introduces into this problem the parameter β. This parameter also leads to a breaking of degeneracy of the energy levels of the oscillator. This parameter is due to the torsion associated with the defect, therefore the torsion provokes the observed break of degeneracy.

6. Concluding remarks

In this work we study the behaviour of a quantum oscillator in the presence of conical singularities. We investigate the quantum dynamics of a single particle interacting harmonically with a conical singularity. The presence of defects in all cases break the degeneracy of the harmonic oscillator. It is suggested that these results may be utilized as a method of detection of cosmic defects and also of defects in solids. This complements recent studies [17–19] on the influence of the geometry and topology of defects on the quantum dynamics of a free particle.

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